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On the Convergence of Solutions of Functional Differential Equations with Infinite Delay

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I. INTRODUCTION

In recent years there has been an increasing interest in infinite delay equations. The main reason is that equations of this type are becoming more and more important for different applications. The theory of existence, uniqueness, continuation of solutions, and continuous dependence on initial data is developed by Driver [2], Hale and Kato [5], Kappel and Schappacher [6]. Driver [2], Grimmer and Seifert [3, 12], Kato [7], Terjéki [13] discuss the stability theory for functional differential equations with infinite delay. The question of convergence of solutions is also appropriate for delay differential equations where any constant is a solution, since for this case, asymptotic stability of any solution is impossible. In [10, 11] Parrott gives sufficient conditions for the convergence of solutions of infinite delay differential equations. These methods are not applicable, when the right-hand side of the equation is the sum of an ordinary and a functional part of the same order. But such equations occur in the applications (see, e.g., [1, 4]).

In this paper conditions are given to guarantee the existence of the limit as $t \to \infty$ of Liapunov functions along the solutions of functional differential equations with infinite delay. The proof of the main theorem is based on differential inequalities. We work in a general state space, for which the fundamental theories have been developed in [5, 6]. Applications are given for C_γ and $\mathscr{L}_k^p$ (defined in Sect. 2). Our result can be applied to prove the convergence of solutions even if the ordinary and functional part of the right-hand side of the equation have the same order.

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2. Notations and Definitions

Let R^n, R, R^+, R^- be the n-dimensional vector space, the set of real numbers, the set of nonnegative and nonpositive real numbers, respectively, and let $|\cdot|$ denote a norm in R^n. Let B be a real vector space of R^n-valued functions defined on R^-, which is endowed with a semi-norm $|\cdot|_B$. If $x: (-\infty, A) \to R^n$, $A \in R$, is a given function, let $x_t: R^- \to R^n$, for each $t \in (-\infty, A)$, be defined by $x_t(s) = x(t+s)$, $s \in R^-$.

Consider the functional differential equation with infinite delay

$$x'(t) = F(t, x_t), \tag{1}$$

where $F: R^+ \times B \to R^n$.

Let $(t_0, \varphi) \in R^+ \times B$ be given. A function $x(\cdot)$ is said to be a solution of (1) on $[t_0, t_0 + A)$ through (t_0, φ), where $0 < A \leqslant \infty$, if $x(\cdot)$ is defined on $(-\infty, t_0 + A)$, absolutely continuous on bounded subintervals of $[t_0, t_0 + A)$, $(t, x_t) \in R^+ \times B$ for almost all $t \in [t_0, t_0 + A)$, $x_{t_0} = \varphi$ and (1) is satisfied almost everywhere on $[t_0, t_0 + A)$.

Denote by $x(t_0, \varphi)$ any noncontinuable solution of (1) through (t_0, φ) and by $x(t_0, \varphi)(t)$ its value at t.

In this paper we suppose that the function F and the vector space B have properties to guarantee the existence of $x(t_0, \varphi)$ on $[t_0, \infty)$. In [5, 6] sufficient conditions for F and B are given for the existence and continuation of solutions. For example, the spaces $C(-r, 0)$, C_γ, $\mathscr{L}_k^p$ satisfy these axioms, where

(i) $C(-r, 0)$, $r > 0$, is the space of functions $\varphi: R^- \to R^n$ that are continuous on $[-r, 0)$ with the semi-norm

$$|\varphi|_{C(-r,0)} = \sup_{-r \leqslant s \leqslant 0} |\varphi(s)|.$$

(ii) C_γ, $\gamma \in R$, denotes the space of continuous functions $\varphi: R^- \to R^n$ having the limit $\lim_{s \to -\infty} e^{\gamma s} \varphi(s)$ with norm

$$|\varphi|_{C_\gamma} = \sup_{s \leqslant 0} e^{\gamma s} |\varphi(s)|.$$

(iii) Suppose that $1 \leqslant p < \infty$ and $k: R^- \to R^+$ is a function so that k is locally integrable over R^- and for all $t > 0$

$$\operatorname*{ess.\,sup}_{-t \leqslant s \leqslant 0} k(s) < \infty,$$

and there exists a function $K: R^- \to R^+$ such that for almost all $s \in R^-$ and all $t \in R^-$,

$$k(t+s) \leqslant K(t)\, k(s).$$

$\mathscr{L}_k^p$ is the vector space of all measurable functions $\varphi: R^- \to R^n$ such that

$$\int_{-\infty}^0 |\varphi(s)|^p k(s)\, ds < \infty$$

with the semi-norm

$$|\varphi|_{\mathscr{L}_k^p} = \left(|\varphi(0)|^p + \int_{-\infty}^0 |\varphi(s)|^p k(s)\, ds \right)^{1/p}.$$

By a Liapunov function we mean a continuous, locally Lipschitzian function $V: R^+ \times R^n \to R^+$. The upper right-hand derivative $D_{(1)}^+ V$ of a Liapunov function V with respect to system (1) is defined by

$$D_{(1)}^+ V(t, \varphi) = \overline{\lim_{\delta \to 0_+}} \frac{1}{\delta} [V(t+\delta, \varphi(0) + \delta F(t, \varphi)) - V(t, \varphi(0))]$$

$$((t, \varphi) \in R^+ \times B).$$

We remark that if $x(\cdot) = x(t_0, \varphi)(\cdot)$ is a solution of (1) on $[t_0, t_0 + A)$, then $V(\cdot, x(\cdot))$ is absolutely continuous on bounded subintervals of $[t_0, t_0 + A)$ and

$$D_{(1)}^+ V(t, x_t) = V'(t, x(t))$$

a.e. on $[t_0, t_0 + A)$.

For a Liapunov function V and $\varphi \in B$, $0 \leqslant t_1 \leqslant t_2$, define the functional W as follows

$$W(t_1, t_2, \varphi) = \sup_{t_1 - t_2 \leqslant s \leqslant 0} V(t_2 + s, \varphi(s)).$$

For a Liapunov function V and given numbers $\varepsilon > 0$, $\eta > 0$, $0 \leqslant t_1 \leqslant t_2$, let

$$\begin{aligned} &S(V, \varepsilon, \eta, t_1, t_2) \\ &\quad = \{\varphi \in B: \varphi_{t_1 - t_2} \in B,\ V(t_2, \varphi(0)) \geqslant \varepsilon, \\ &\quad\quad W(t_1, t_2, \varphi) \leqslant 2\varepsilon,\ W(t_1, t_2, \varphi) - V(t_2, \varphi(0)) < \eta\}. \end{aligned}$$

3. The Main Result

The main result guarantees the existence of the limit of a Liapunov function along solutions of (1) as $t \to \infty$.

We first give a simple differential inequality, which is a special case of a result of Lakshmikantham and Leela [9, Theorem 1.10.2].

LEMMA. *If $u: R^+ \to R^+$ is absolutely continuous on bounded subintervals of R^+, $p, q: R^+ \to R^+$, $p, q \in L^1(R^+)$, $A \in R$, $T \in R^+$ and*

$$u'(t) \leqslant [A - u(t)]\, p(t) + q(t) \qquad (\text{a.e. on } [T, \infty)),$$
$$u(T) = u_0,$$

then

$$u(t) \leqslant A + (u_0 - A) \exp\left(-\int_T^t p(s)\, ds\right) + \int_T^t q(s) \exp\left(-\int_s^t p(r)\, dr\right) ds \qquad (t \geqslant T).$$

THEOREM 1. *Suppose that for a Liapunov function V there exist functions $\alpha: R^+ \times R^+ \times (0, \infty) \to R^+$, $\beta: R^+ \times R^+ \times B \to R^+$, $\gamma: (0, \infty) \to (0, \infty)$, $\eta: (0, \infty) \to (0, \infty)$, and a number $r \in R^+$ such that*

(i) *for every $u \geqslant 0$, $v > 0$, and $\varphi \in B$,*

$$\int_u^\infty \alpha(t, u, v)\, dt \leqslant \gamma(v), \tag{2}$$

$$\int_u^\infty \beta(t, u, \varphi)\, dt < \infty; \tag{3}$$

(ii) *if $\varepsilon > 0$, $0 \leqslant t_1 \leqslant t_1 + r \leqslant t_2 \leqslant t$, and*

$$\varphi \in S(V, \varepsilon, \eta(\varepsilon), t_1, t), \qquad V(t, \varphi(0)) = W(t_2, t, \varphi), \tag{4}$$

then

$$D^+_{(1)} V(t, \varphi) \leqslant [W(t_1, t, \varphi) - V(t, \varphi(0))]\, \alpha(t, t_2, \varepsilon) + \beta(t, t_1, \varphi_{t_1 - t}). \tag{5}$$

Then for each $(t_0, \varphi) \in R^+ \times B$ the limit $\lim_{t \to \infty} V(t, x(t_0, \varphi)(t))$ exists.

Proof. Let $x(t_0, \varphi)(\cdot)$ be a solution of (1) on $[t_0, \infty)$, and let $v(t) = V(t, x(t_0, \varphi)(t))$, $t \geqslant t_0$. We first prove that v is bounded on $[t_0, \infty)$. Let

$$u_1(t) = \max\{1, \max_{t_0 \leqslant s \leqslant t} v(s)\} \qquad (t \geqslant t_0),$$

$$S_1 = \{t \in [t_0 + r, \infty): u_1(t) = v(t)\}.$$

Clearly u_1 is absolutely continuous on bounded subintervals of $[t_0, \infty)$. It is easy to see that if $t \in [t_0 + r, \infty) \setminus S_1$, then $u_1'(t) = 0$ and

$u_1'(t) \leqslant \max\{v'(t), 0\}$, a.e. on S_1. Apply condition (ii) setting $t \in S_1$, $t_1 = t_0$, $t_2 = t$, $\varepsilon = v(t_2)$. We obtain

$$u_1'(t) \leqslant \max\{\beta(t, t_0, x_{t_0}), 0\} = \beta(t, t_0, \varphi) \qquad (\text{a.e. on } S_1).$$

Since β is nonnegative

$$u_1'(t) \leqslant \beta(t, t_0, \varphi) \qquad (\text{a.e. on } [t_0 + r, \infty)).$$

Hence, by using the lemma and condition (3), it follows that u_1 is bounded on $[t_0, \infty)$. This implies that v is also bounded on $[t_0, \infty)$.

Assume that the limit $\lim_{t \to \infty} v(t)$ does not exist. Then there are numbers ε, a, b such that $0 < \varepsilon \leqslant b < a < 2\varepsilon$, $a - b < \eta(\varepsilon)$, $\overline{\lim}_{t \to \infty} v(t) = a$, $\underline{\lim}_{t \to \infty} v(t) < b$. Let δ, t_1 be defined such that

$$0 < \delta < \min\left\{\frac{(a-b)\, e^{-\gamma(\varepsilon)}}{2(1 - e^{-\gamma(\varepsilon)})}, \eta(\varepsilon) - (a - b)\right\}, \tag{6}$$

$t_1 \geqslant t_0$ and $v(t) \leqslant a + \delta$ for $t \geqslant t_1$. By using condition (3), we have $\beta(\cdot, t_1, x_{t_1}) \in L^1([t_1, \infty))$. From this and $\underline{\lim}_{t \to \infty} v(t) < b$ it follows that there exists a number t_2 such that $t_2 \geqslant t_1 + r$, $v(t_2) = b$ and

$$\int_{t_2}^{\infty} \beta(s, t_1, x_{t_1})\, ds < \frac{(a-b)\, e^{-\gamma(\varepsilon)}}{2}. \tag{7}$$

Define

$$u_2(t) = \max_{t_2 \leqslant s \leqslant t} v(s) \qquad (t \geqslant t_2),$$

$$S_2 = \{t \in [t_2, \infty) \colon u_2(t) = v(t)\}.$$

Clearly u_2 is absolutely continuous on bounded intervals of $[t_2, \infty)$. It is easy to verify that if $t \in [t_2, \infty) \backslash S_2$, then $u_2'(t) = 0$, and $u_2'(t) \leqslant \max\{v'(t), 0\}$ a.e. on S_2. Obviously

$$x_t \in S(V, \varepsilon, \eta(\varepsilon), t_1, t), \qquad v(t) = \max_{t_2 \leqslant s \leqslant t} v(s) \qquad (t \in S_2).$$

Thus, by condition (ii), $u_2(t) \leqslant a + \delta$ for $t \geqslant t_1$, and the nonnegativity of α, β, we have

$$u_2'(t) \leqslant [a + \delta - u_2(t)]\, \alpha(t, t_2, \varepsilon) + \beta(t, t_1, x_{t_1}) \tag{8}$$

a.e. on S_2. From the nonnegativity of the right-hand side of (8) it follows that relation (8) is also valid a.e. on $[t_2, \infty)$. Apply the lemma setting

$u(\cdot)=u_2(\cdot)$, $p(\cdot)=\alpha(\cdot, t_2, \varepsilon)$, $q(\cdot)=\beta(\cdot, t_1, x_{t_1})$, $A=a+\delta$, $T=t_2$, $u_0=b$. We obtain

$$u_2(t)\leqslant a+\delta+(b-a-\delta)\exp\left(-\int_{t_2}^{t}\alpha(s, t_2, \varepsilon)\,ds\right)$$
$$+\int_{t_2}^{t}\beta(s, t_1, x_{t_1})\exp\left(-\int_{s}^{t}\alpha(\tau, t_2, \varepsilon)\,d\tau\right)ds \qquad (t\geqslant t_2). \tag{9}$$

Inequalities (2), (6), (7), and (9) imply that

$$u_2(t)\leqslant a+\delta+(b-a-\delta)\,e^{-\gamma(\varepsilon)}+\frac{(a-b)\,e^{-\gamma(\varepsilon)}}{2}<a \qquad (t\geqslant t_2). \tag{10}$$

From (10) and the definition of u_2 we get $\overline{\lim}_{t\to\infty} v(t)<a$, which is a contradiction. This completes the proof.

Theorem 1 implies easily a corollary for the finite delay case (i.e. $B=C(-r, 0)$), which is the main result of [8].

COROLLARY. *Let $r>0$ and $B=C(-r, 0)$. Suppose that for a Liapunov function V there exist functions ξ, η: $(0, \infty)\to(0, \infty)$ such that if $\varepsilon>0$, $t\geqslant 0$, and $\varphi\in S(V, \varepsilon, \eta(\varepsilon), t-r, t)$, then*

$$D^+_{(1)}V(t, \varphi)\leqslant\xi(\varepsilon)[W(t-r, t, \varphi)-V(t, \varphi(0))].$$

Then for each $(t_0, \varphi)\in R^+\times C(-r, 0)$ the limit $\lim_{t\to\infty} V(t, x(t_0, \varphi)(t))$ *exists.*

Remark 1. If $\lim_{|x|\to\infty} V(t, x)=\infty$ for every $t\in R^+$ in addition to the assumptions of Theorem 1, and the existence of $x(t_0, \varphi)$ is not assumed on $[t_0, \infty)$, but local existence is on some interval $[t_0, t_0+A)$, $0<A\leqslant\infty$, and $A<\infty$ implies $\overline{\lim}_{t\to t_0+A^-} |x(t_0, \varphi)(t)|=\infty$, then it can also be proved that any solution $x(t_0, \varphi)$ of (1), $(t_0, \varphi)\in R^+\times B$, is continuable on $[t_0, \infty)$.

4. APPLICATIONS AND EXAMPLES

I. Consider the equation

$$x'(t)=f(t, x(t))+g(t, x_t), \tag{11}$$

where f: $R^+\times R^n\to R^n$, g: $R^+\times B\to R^n$ are continuous functions. Equation (11) is a special form of (1), when $F(t, \varphi)=f(t, \varphi(0))+g(t, \varphi)$.

THEOREM 2. *Suppose that the functions* $h: R^+ \times R^+ \to R^+$, $L: (0, \infty) \times R^+ \to R^+$, $p, q: R^+ \to R^+$, *and* $\mu: R^+ \times R^- \to [0, 1]$ *have the following properties*:

(i) $h(t, \cdot)$ *is nondecreasing, for each* $\varepsilon > 0$ *the function* $h(t, \cdot)$ *is Lipschitzian on* $[\varepsilon, 2\varepsilon]$ *with the Lipschitz constant* $L(\varepsilon, t)$;

(ii) $\mu(t, \cdot)$ *is nondecreasing and continuous from the right on* R^-, $\lim_{s \to -\infty} \mu(t, s) = 0$, *and for all* $u \geqslant 0$, $v > 0$,

$$\int_u^\infty L(v, t)\, \mu(t, u-t)\, dt \leqslant q(v); \tag{12}$$

(iii) $p \in L^1(R^+)$.

If for every $t, u \in R^+$, $x \in R^n$, $\varphi \in B$,

$$\overline{\lim_{\delta \to 0+}} \frac{1}{\delta} [|x + \delta f(t, x)| - |x|] \leqslant -h(t, |x|), \tag{13}$$

$$|g(t, \varphi)| \leqslant \int_{-\infty}^0 h(t, |\varphi(s)|)\, d_s\mu(t, s) + p(t), \tag{14}$$

$$\int_u^\infty \int_{-\infty}^{u-t} h(t, |\varphi(s+t-u)|)\, d_s\mu(t, s)\, dt < \infty, \tag{15}$$

then for each solution $x(t_0, \varphi)(\cdot)$ *of* (11), $(t_0, \varphi) \in R^+ \times B$, *the limit* $\lim_{t \to \infty} |x(t_0, \varphi)(t)|$ *exists.*

Proof. Apply Theorem 1 setting $V(t, x) = |x|$,

$$\alpha(t, u, v) = L(v, t)\, \mu(t, u-t),$$

$$\beta(t, u, \varphi) = \int_{-\infty}^{u-t} h(t, |\varphi(s+t-u)|)\, d_s\mu(t, s) + p(t),$$

$\gamma(u) = q(u)$, $\eta(u) = u$, and $r = 0$. Inequalities (12) and (15) and property (iii) imply that α, β, γ satisfy condition (i) of Theorem 1. Using properties (i), (ii), (iii) we can check condition (ii) of Theorem 1 as follows: if $\varepsilon > 0$, $0 \leqslant t_1 \leqslant t_2 \leqslant t$, $\varphi \in S(V, \varepsilon, \varepsilon, t_1, t)$, and $|\varphi(0)| = \sup_{t_2 - t \leqslant s \leqslant 0} |\varphi(s)|$, then

$$D^+_{(1)} V(t, \varphi)$$

$$= \overline{\lim_{\delta \to 0+}} \frac{1}{\delta} [|\varphi(0) + \delta f(t, \varphi(0)) + \delta g(t, \varphi)| - |\varphi(0)|]$$

$$\leqslant \overline{\lim_{\delta \to 0+}} \frac{1}{\delta} [|\varphi(0) + \delta f(t, \varphi(0))| - |\varphi(0)|] + |g(t, \varphi)|$$

$$\leqslant -h(t, |\varphi(0)|) + \int_{-\infty}^{0} h(t, |\varphi(s)|) \, d_s\mu(t, s) + p(t)$$

$$\leqslant -\int_{-\infty}^{0} h(t, |\varphi(0)|) \, d_s\mu(t, s) + \int_{t_2 - t}^{0} h(t, |\varphi(s)|) \, d_s\mu(t, s)$$

$$+ \int_{t_1 - t}^{t_2 - t} h(t, |\varphi(s)|) \, d_s\mu(t, s) + \int_{-\infty}^{t_1 - t} h(t, |\varphi(s)|) \, d_s\mu(t, s) + p(t)$$

$$\leqslant \int_{t_1 - t}^{t_2 - t} [h(t, \sup_{t_1 - t \leqslant u \leqslant 0} |\varphi(u)|) - h(t, |\varphi(0)|)] \, d_s\mu(t, s)$$

$$+ \int_{-\infty}^{t_1 - t} h(t, |\varphi(s)|) \, d_s\mu(t, s) + p(t)$$

$$\leqslant [\sup_{t_1 - t \leqslant u \leqslant 0} |\varphi(u)| - |\varphi(0)|] \, L(\varepsilon, t) \, \mu(t, t_2 - t)$$

$$+ \int_{-\infty}^{t_1 - t} h(t, |\varphi(s)|) \, d_s\mu(t, s) + p(t).$$

This completes the proof.

II. Consider the scalar equation

$$x'(t) = -a(t) \, x(t) + b(t) \, x(t - r(t)), \tag{16}$$

where $a, b \colon R^+ \to R$, $r \colon R^+ \to R^+$ are continuous functions.

THEOREM 3. *Suppose that*

(i) $|b(t)| \leqslant a(t)$ *for* $t \geqslant 0$,

(ii) $\lim_{t \to \infty} [t - r(t)] = \infty$,

(iii) $\overline{\lim}_{u \to \infty} \int_{u - r(u)}^{u} |b(t)| \, dt < \infty$,

(iv) *for every* $u \geqslant 0$, $\varphi \in B$,

$$\int_{A(u)} |b(t)| \cdot |\varphi(t - r(t) - u)| \, dt < \infty,$$

where $A(u) = \{s \colon s - r(s) \leqslant u \leqslant s\}$. *Then for each* $(t_0, \varphi) \in R^+ \times B$ *the solution* $x(t_0, \varphi)$ *of* (16) *tends to a constant as* $t \to \infty$.

Proof. Clearly we can apply Theorem 2 by putting $f(t, x) = -a(t)x$,

$g(t, \varphi) = b(t)\,\varphi(-r(t))$, $\quad h(t, x) = |b(t)|\, x$, $\quad L(v, t) = |b(t)|$, $\quad p(\cdot) \equiv 0$, $q(v) = \sup_{u \geqslant 0} \int_{A(u)} |b(t)|\, dt$,

$$\mu(t, s) = \begin{cases} 0 & \text{if } \; s < -r(t), \\ 1 & \text{if } \; -r(t) \leqslant s \leqslant 0, \end{cases}$$

and $R^n = R$.

Remark 2. For example, the space C_γ, $\gamma \in R$, satisfies condition (iv) of Theorem 3 for any continuous function $b(t)$.

In [2, 7, 11], Eq. (16) is studied provided that $|b(t)| \leqslant a(t)\Theta$, $t \geqslant 0$, $0 \leqslant \Theta < 1$, $\lim_{t \to \infty} [t - r(t)] = \infty$. In the case $a(\cdot) \equiv a > 0$ asymptotic stability is proved by Driver [2], and in the case $a(\cdot) \equiv a > 0$, $\varepsilon t - N \leqslant t - r(t) \leqslant t$, $t \geqslant 0$, $\varepsilon > 0$, equi-asymptotic stability is proved by Kato [7] for the zero solution of (16), considered as an equation on $R^+ \times C_\gamma$, $\gamma > 0$. Parrott shows that each solution $x(t_0, \varphi)$ of (16), $(t_0, \varphi) \in R^+ \times BC((-\infty, 0], R)$, tends to a constant as $t \to \infty$, and if $\int_0^\infty a(t)\, dt = \infty$, then the zero solution is asymptotically stable. $BC((-\infty, 0], R)$ denotes the space of bounded continuous functions mapping $(-\infty, 0]$ into R. Clearly $BC \subset C_\gamma$ for any $\gamma > 0$.

Conditions of Theorem 3 and $\int_0^\infty a(t)\, dt = \infty$ do not imply convergence to zero, since any constant is a solution of (16), when $b(t) = a(t)$, $t \geqslant 0$.

Assumption (iii) of Theorem 3 cannot be omitted. We show that there exist continuous functions x, r, φ such that $r(t) \geqslant 0$, $t \geqslant 0$, $\lim_{t \to \infty} [t - r(t)] = \infty$, $\varphi \in C_0$, $x(s) = \varphi(s)$, $s \leqslant 0$,

$$x'(t) = -x(t) + x(t - r(t)) \qquad (t \geqslant 0) \tag{17}$$

and $\overline{\lim}_{t \to \infty} x(t) > \underline{\lim}_{t \to \infty} x(t)$. Let

$$\varphi(s) = \begin{cases} 0 & \text{if } \; s \leqslant -1, \\ s + 1 & \text{if } \; -1 < s \leqslant 0. \end{cases}$$

Functions x, r and a sequence $\{t_n\}$ can be constructed by the method of steps as follows: let $t_{-2} = t_{-1} = -1$, $t_0 = 0$, $x(s) = \varphi(s)$, $s \leqslant 0$, and suppose that $t_1, t_2, \ldots, t_{4k}$ and $x(t)$, $r(t)$, $0 \leqslant t \leqslant t_{4k}$, are defined so that x is a solution of (17) through $(0, \varphi)$ on $[0, t_{4k}]$ and $t_{4k} - r(t_{4k}) = t_{4k-2}$. Define t_{4k+1}, $x(t)$, $r(t)$, $t_{4k} \leqslant t \leqslant t_{4k+1}$: $t_{4k+1} > t_{4k}$, $t - r(t) = t_{4k-2}$ for $t \in [t_{4k}, t_{4k+1}]$, x is a solution of (17) on $[t_{4k}, t_{4k+1}]$ and $x(t_{4k+1}) \leqslant x(t_{4k-2}) + (1/2^{4k+2})$. Let t_{4k+2}, $x(t)$, $r(t)$ for $t \in [t_{4k+1}, t_{4k+2}]$ be chosen such that $t_{4k+2} > t_{4k+1}$, $t_{4k+2} - r(t_{4k+2}) = t_{4k}$, x is a solution of (17) on $[t_{4k+1}, t_{4k+2}]$ and $x(t_{4k+2}) \leqslant x(t_{4k+1}) + (1/2^{4k+3})$. Let $t_{4k+3} > t_{4k+2}$ be so that $t - r(t) = t_{4k}$ for $t \in [t_{4k+2}, t_{4k+3}]$, x is a solution of (17) on $[t_{4k+2}, t_{4k+3}]$, and $x(t_{4k+3}) \geqslant x(t_{4k}) - (1/2^{4k+4})$. Define t_{4k+4}, $x(t)$, $r(t)$ for $t \in [t_{4k+3}, t_{4k+4}]$, as follows: $t_{4k+4} > t_{4k+3}$, $t_{4k+4} - r(t_{4k+4}) = t_{4k+2}$, x is a solution of (17) for $t \in [t_{4k+3}, t_{4k+4}]$ and $x(t_{4k+4}) \geq x(t_{4k+3}) - (1/2^{4k+5})$. It is easy to see that $t_{4k+4} - t_{4k} \to \infty$ as $k \to \infty$ and condition (iii) of Theorem 3 is not satisfied. $\sum_{k=0}^{\infty} ((1/2^{4k+2}) + (1/2^{4k+3}) + (1/2^{4k+4}) + (1/2^{4k+5})) = \frac{1}{2}$ implies $\overline{\lim}_{t\to\infty} x(t) > \underline{\lim}_{t\to\infty} x(t)$.

III. Let us consider the scalar equation

$$x'(t) = -a(t)\, x^{\alpha}(t) + \int_{-\infty}^{0} b(t, s, x(t+s))\, ds + c(t), \tag{18}$$

where a, c: $R^+ \to R$, b: $R^+ \times R^- \times R \to R$ are continuous functions, $\alpha > 0$ is the quotient of odd integers.

In the next two theorems sufficient conditions are given for the convergence of solutions of (18) in the state spaces $R^+ \times C_\gamma$ and $R^+ \times \mathscr{L}_k^p$.

THEOREM 4. *Assume that*

(i) *there exists a function* m: $R^+ \times R^- \to R^+$ *and a positive number* γ *such that*

$$|b(t, s, r)| \leq m(t, s)\, |r|^{\alpha} \qquad ((t, s, r) \in R^+ \times R^- \times R), \tag{19}$$

$$\int_{-\infty}^{0} m(t, s)\, ds \leq a(t) \qquad (t \in R^+), \tag{20}$$

$$\int_{u}^{\infty} \int_{-\infty}^{u-t} m(t, s)\, e^{\alpha\gamma(u-t-s)}\, ds\, dt \leq K \qquad (u \geq 0). \tag{21}$$

(ii) $c \in L^1(R^+)$.

Then for each solution $x(t_0, \varphi)$ *of* (18), *considered as an equation on* $[t_0, \infty) \times C_\gamma$, $(t_0, \varphi) \in R^+ \times C_\gamma$, *the limit* $\lim_{t\to\infty} x(t_0, \varphi)(t)$ *exists.*

Proof. Apply Theorem 2 setting $R^n = R$, $B = C_\gamma$,

$$f(t, x) = -a(t)\, x^{\alpha}, \qquad g(t, \varphi) = \int_{-\infty}^{0} b(t, s, \varphi(s))\, ds + c(t),$$

$$h(t, x) = \left(\int_{-\infty}^{0} m(t, s)\, ds\right) x^{\alpha}, \qquad p(t) = |c(t)|,$$

$$L(v, t) = \left(\int_{-\infty}^{0} m(t, s)\, ds\right) \alpha \cdot \max\{x^{\alpha-1} \colon v \leq x \leq 2v\},$$

$$q(v) = K\alpha \cdot \max\{x^{\alpha-1} \colon v \leq x \leq 2v\}$$

and

$$\mu(t,s)=\begin{cases}\dfrac{\int_{-\infty}^{s} m(t,u)\,du}{\int_{-\infty}^{0} m(t,u)\,du} & \text{if } \displaystyle\int_{-\infty}^{0} m(t,u)\,du\neq 0,\\[2ex] 0 & \text{if } \displaystyle\int_{-\infty}^{0} m(t,u)\,du=0.\end{cases}$$

Inequality (21) implies for $u\geqslant 0$, $v>0$,

$$\begin{aligned}&\int_u^\infty L(v,t)\,\mu(t,u-t)\,dt\\&\quad=\alpha\int_u^\infty\int_{-\infty}^{u-t} m(t,s)\,ds\,dt\cdot\max\{x^{\alpha-1}: v\leqslant x\leqslant 2v\}\\&\quad\leqslant\alpha\int_u^\infty\int_{-\infty}^{u-t} m(t,s)\,e^{\alpha\gamma(u-t-s)}\,ds\,dt\cdot\max\{x^{\alpha-1}: v\leqslant x\leqslant 2v\}\\&\quad\leqslant\alpha K\cdot\max\{x^{\alpha-1}: v\leqslant x\leqslant 2v\}.\end{aligned}$$

If $\varphi\in C_\gamma$, then $|\varphi(s)|\leqslant K_\varphi e^{-\gamma s}$, $s\leqslant 0$, for some positive constant K_φ. Hence and from (21)

$$\begin{aligned}&\int_u^\infty\int_{-\infty}^{u-t} h(t,|\varphi(s+t-u)|)\,d_s\mu(t,s)\,dt\\&\quad\leqslant K_\varphi^\alpha\int_u^\infty\int_{-\infty}^{u-t} m(t,s)\,e^{\alpha\gamma(u-t-s)}\,ds\,dt\leqslant K_\varphi^\alpha\cdot K\qquad(u\geqslant 0).\end{aligned}$$

Thus, clearly conditions (i), (ii), (iii) of Theorem 2 and (13), (14), (15) are satisfied.

Remark 3. Kato [7] proves that if $a(\cdot)\equiv a>0$, $c(\cdot)\equiv 0$, $\alpha=1$, $|b(t,s,r)|\leqslant m(s)|r|$, $\int_{-\infty}^{0} m(s)\,ds<a$, and $\int_{-\infty}^{0} m(s)\,e^{-\gamma s}\,ds<\infty$ for a $\gamma>0$, then the zero solution of (18), considered as an equation on $R^+\times C_\gamma$, is uniformly asymptotically stable.

Parrott [10] shows the convergence of solutions of (18), considered as an equation on $R^+\times C_\gamma$, $\gamma>0$, provided that $\alpha=1$, $|b(t,s,r)|\leqslant m(t,s)|r|$ and $\int_{-\infty}^{0} m(t,s)\,e^{-\gamma s}\,ds\leqslant a(t)\Theta$, $0\leqslant\Theta<1$.

In Kato and Parrott's results the constant functions cannot be solutions of (18), except for the trivial case $a(\cdot)\equiv 0$ in [10]. If in (20) equality holds, then conditions (19), (20), (21) allow constants to be solutions of (18) when $c(t)\equiv 0$. Thus, asymptotic stability of any solution is impossible.

THEOREM 5. *Suppose that*

(i) *there exist functions* $m\colon R^+ \to R^+$, $k, K\colon R^- \to R^+$ *such that* $\int_{-\infty}^{0} k(s)\,ds = 1$, $\operatorname{ess\,sup}_{-t \leqslant s \leqslant 0} k(s) < \infty$ *for all* $t > 0$, *and for almost all* $s \in R^-$, *all* $t \in R^-$,

$$k(t+s) \leqslant K(t)\,k(s), \tag{22}$$

$$|b(t,s,r)| \leqslant m(t)\,k(s)|r|^\alpha \qquad ((t,s,r) \in R^+ \times R^- \times R),$$

$$m(t) \leqslant a(t) \qquad (t \in R^+),$$

$$\int_u^\infty \int_{-\infty}^{u-t} m(t)\,k(s)\,ds\,dt \leqslant M \qquad (u \geqslant 0),$$

$$\int_u^\infty m(t)\,K(u-t)\,dt < \infty \qquad (u \geqslant 0). \tag{23}$$

(ii) $c \in L^1(R^+)$.

Then every solution of (18), *considered as an equation on* $R^+ \times \mathscr{L}_k^p$, $p \geqslant \max\{\alpha, 1\}$, *tends to a constant as* $t \to \infty$.

Proof. Clearly our statement follows from Theorem 2 setting $f(t,x) = -a(t)\,x^\alpha$, $g(t,\varphi) = \int_{-\infty}^{0} b(t,s,\varphi(s))\,ds$, $h(t,x) = m(t)\,x^\alpha$, $p(t) = |c(t)|$, $L(v,t) = m(t)\,\alpha \max\{x^{\alpha-1}\colon v \leqslant x \leqslant 2v\}$, $q(v) = M\alpha \max\{x^{\alpha-1}\colon v \leqslant x \leqslant 2v\}$, $\mu(t,s) = \int_{-\infty}^{s} k(u)\,du$, $R^n = R$, and $B = \mathscr{L}_k^p$. It is easy to see that conditions (i), (ii), (iii) of Theorem 2 and (13), (14) are satisfied in this case. Inequality (15) can be proved by using (22), (23), the Hölder's inequality, and the definition of $\mathscr{L}_k^p$ as follows:

$$\begin{aligned}
&\int_u^\infty \int_{-\infty}^{u-t} h(t, |\varphi(s+t-u)|)\,d_s\mu(t,s)\,dt \\
&\quad = \int_u^\infty m(t) \int_{-\infty}^{u-t} |\varphi(s+t-u)|^\alpha\,k(s)\,ds\,dt \\
&\quad = \int_u^\infty m(t) \int_{-\infty}^{0} |\varphi(s)|^\alpha\,k(s+u-t)\,ds\,dt \\
&\quad \leqslant \int_u^\infty m(t)\,K(u-t) \int_{-\infty}^{0} |\varphi(s)|^\alpha [k(s)]^{\alpha/p} [k(s)]^{1-(\alpha/p)}\,ds\,dt \\
&\quad \leqslant |\varphi|_{\mathscr{L}_k^p}^\alpha \int_u^\infty m(t)\,K(u-t)\,dt < \infty \qquad (u \geqslant 0).
\end{aligned}$$

This completes the proof.

EXAMPLE. Consider the equation

$$x'(t) = -x^3(t) + \int_{-\infty}^{0} e^s x^3(t+s)\,ds. \tag{24}$$

An easy application of Theorems 4 and 5 gives that each solution of (24) tends to a constant in the state spaces C_γ for $0 < \gamma < \frac{1}{3}$ and $\mathscr{L}^p_{e^s}$ for $p \geqslant 3$.

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